CONSTRAINED SIMILARITY OF 2-D TRAJECTORIES BY MINIMIZING THE H^1 SEMI-NORM OF THE TRAJECTORY DIFFERENCE

PhD Student Filipov S., Assoc. Prof. Atanassov A., Senior Lecturer Gospodinov I.
Department of Computer Science – University of Chemical Technology and Metallurgy, Bulgaria
naso@uctm.edu

Abstract: *This paper defines constrained functional similarity between 2-D trajectories via minimizing the H^1 semi-norm of the difference between the trajectories. An exact general solution is obtained for the case wherein the components of the trajectories are mesh-functions defined on a uniform mesh and the imposed constraints are linear. Various examples are presented, one of which features application to mechanics and two-point boundary value problems. A MATLAB code is given for the solution of one of the examples. The code could easily be adjusted to other cases.*

Keywords: SIMILARITY OF TRAJECTORIES, H^1 SEMI-NORM MINIMIZATION

1. Introduction

Suppose a trajectory is given and a new trajectory is sought that meets a number of imposed constraints and is as similar in behaviour to the original trajectory as possible without necessarily being close [1] to it. Such shape optimisation problems may have wide range of applications in many engineering fields [2] such as mechanics, fluid mechanics, aerodynamics, general transport phenomena, design and engineering of machines and equipment, etc. In [3] the authors have introduced constrained functional similarity between real-valued functions of one real variable via minimizing the H^1 semi-norm [4] of the difference between the functions. An exact general solution for mesh-functions has been presented. The similarity of trajectories in two and more dimensions is as important. This work defines constrained similarity between 2-D trajectories and provides an exact solution to the discretized case. Application to mechanics and two-point boundary value problems [5] is presented the Results section.

2. Constrained similarity of 2-D trajectories

Let $r^*(t)=(x^*(t),y^*(t))$ and $r(t)=(x(t),y(t))$ be two radius vectors whose components are real-valued functions of a real independent variable $t\in[a,b]$. The functions r^* and r define two 2-D trajectories. The trajectory r^* will be *similar* to r, under certain given constraints, if r^* minimizes the square of the H^1 semi-norm of the difference r^*-r:

$$|r^*-r|^2_{H_1}=\int_a^b\left(\frac{dr^*}{dt}-\frac{dr}{dt}\right)^2dt=\int_a^b\left(\frac{dx^*}{dt}-\frac{dx}{dt}\right)^2dt+\int_a^b\left(\frac{dy^*}{dt}-\frac{dy}{dt}\right)^2dt \quad (1)$$

and at the same time satisfies the constrains in question. The constraints that r^* satisfies must be linear in x^* and y^*. For example, linear combinations of functional values $x^*(t_i)$ and $y^*(t_i)$ at certain points t_i, integral constraints like

$\int_a^b f(t)x^*(t)dt=1$ or $\int_a^b g(t)y^*(t)dt=1$, etc.

3. Exact solution for discretized trajectories under linear constraints

Partitioning the interval $t\in[a,b]$ by N mesh points into $N-1$ intervals of equal size defines a uniform mesh on the interval: $\{t_i=a+(i-1)h,\ i=1,2,\dots,N,\ h=(b-a)/(N-1)\}$, where h is the step-size of the mesh. Let the trajectory r be defined on the mesh, i.e. $\{r_i=r(t_i),\ i=1,2,\dots,N\}$. In order to define constrained similarity between the trajectories r^* and r expression (1) is discretized using the forward finite differences $(x^*_{i+1}-x^*_i)/h$, etc. for the respective derivatives dx^*/dt, etc. at t_i, $i=1,2,\dots,N-1$ and the integral is replaced by a sum. The constant h is omitted because constant factors do not affect the minimization. Thus, the following objective function is obtained:

$$I=\sum_{i=1}^{N-1}((x^*_{i+1}-x^*_i)-(x_{i+1}-x_i))^2+\sum_{i=1}^{N-1}((y^*_{i+1}-y^*_i)-(y_{i+1}-y_i))^2 \quad (2)$$

In order to use the formulas derived in [3] we denote $x_i=u_i$, $y_i=u_{N+i}$, $x^*_i=u^*_i$, and $y^*_i=u^*_{N+i}$, for $i=1,2,\dots,N$ and introduce the two vectors $u=[x_1,\dots,x_N,y_1,\dots,y_N]^T$ and $u^*=[x^*_1,\dots,x^*_N,y^*_1,\dots,y^*_N]^T$. The minimum of I is sought subject to M linear constraints:

$$\sum_{i=1}^{2N}A_{ji}u^*_i=c_j,\qquad j=1,2,\dots,M<2N. \quad (3)$$

The constraints (3) can be written in a matrix form as $Au^*=c$, where

$$A=\begin{bmatrix}A_{11} & A_{12} & . & . & . & A_{1(2N)}\\ A_{21} & A_{22} & . & . & . & A_{2(2N)}\\ . & . & . & . & . & .\\ A_{M1} & A_{M2} & . & . & . & A_{M(2N)}\end{bmatrix},\quad c=\begin{bmatrix}c_1\\ c_2\\ .\\ c_M\end{bmatrix} \quad (4)$$

and u^* is the $2N\times1$ column-vector of the unknowns. To find the minimum of I subject to constraints (3) the Lagrange's method of the undetermined coefficients [6] is used. First, the Lagrangian

$$J=I+\sum_{j=1}^{M}\left(\lambda_j(c_j-\sum_{i=1}^{2N}A_{ji}u^*_i)\right) \quad (5)$$

is introduced, where λ_j, $j=1,2,\dots,M$ are the Lagrange's undetermined coefficients. Then, the derivatives of J with respect to u^*_k, $k=1,2,\dots,2N$ are equated to zero:

$$\frac{\partial J}{\partial u^*_k}=2((u^*_k-u^*_{k-1})-(u_k-u_{k-1}))-2((u^*_{k+1}-u^*_k)-(u_{k+1}-u_k))-\sum_{j=1}^{M}\lambda_jA_{jk}=0,$$
$$k=2,..,N-1,N+2,..,2N-1$$

$$\frac{\partial J}{\partial u^*_1}=-2((u^*_2-u^*_1)-(u_2-u_1))-\sum_{j=1}^{M}\lambda_jA_{j1}=0$$

$$\frac{\partial J}{\partial u^*_N}=2((u^*_N-u^*_{N-1})-(u_N-u_{N-1}))-\sum_{j=1}^{M}\lambda_jA_{jN}=0 \quad (6)$$

$$\frac{\partial J}{\partial u^*_{N+1}}=-2((u^*_{N+2}-u^*_{N+1})-(u_{N+2}-u_{N+1}))-\sum_{j=1}^{M}\lambda_jA_{j(N+1)}=0$$

$$\frac{\partial J}{\partial u^*_{2N}}=2((u^*_{2N}-u^*_{2N-1})-(u_{2N}-u_{2N-1}))-\sum_{j=1}^{M}\lambda_jA_{j(2N)}=0$$

The system of equations (6) is rearranged so that only terms containing u^*_i remain on the left-hand side. Then, the system is written in a matrix form as:

$$\bar{L}u^*=\bar{L}u-\frac{1}{2}A^T\lambda, \quad (7)$$

where λ is the $M\times1$ column-vector of the undetermined coefficients and $\bar{L}$ is the following $2N\times2N$ matrix:

$$\overline{L}=\begin{bmatrix} -1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & -2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & -2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ & & & \ddots & & & & & & & \ddots & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & -2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -2 & 1 & \cdot & \cdot & \cdot & 0 \\ & & & \ddots & & & & & & & \ddots & & & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -2 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -1 \end{bmatrix} \quad (8)$$

In order to remove the singularity of $\overline{L}$ at least one of the equations for the constraints needs to be added to one of the first N equations in (7) and at least one of the equations for the constraints needs to be added to one of the second N equations in (7). For this reason a $2N\times 2N$ matrix $\overline{A}$ is introduced whose first row is any row of the matrix A (say row j) that corresponds to an x-constraint and whose row $N+1$ is any row of the matrix A (say row m) that corresponds to a y-constraint. The rest of the elements of $\overline{A}$ are zeros. A $2N\times 1$ column-vector $\overline{c}$ with only non-zero components $\overline{c}(1)=c(j)$ and $\overline{c}(N+1)=c(m)$ is also introduced. If necessary, more equations from A can be included in $\overline{A}$. Now, the results for u^* and λ, derived in [3], can be used:

$$u^* = u-(\overline{L}+\overline{A})^{-1}\left(\frac{1}{2}A^T\lambda+\overline{A}u-\overline{c}\right), \quad (9)$$

$$\lambda = 2\left(A(\overline{L}+\overline{A})^{-1}A^T\right)^{-1}\left(Au-c-A(\overline{L}+\overline{A})^{-1}(\overline{A}u-\overline{c})\right), \quad (10)$$

where $\overline{L}$ is defined in (8) . The right-hand side of (10) contains only known quantities. Once the column vector λ is calculated it is substituted into (9) and the sought u^*_i, $i=1,2,\dots,2N$ are found.

4. Results

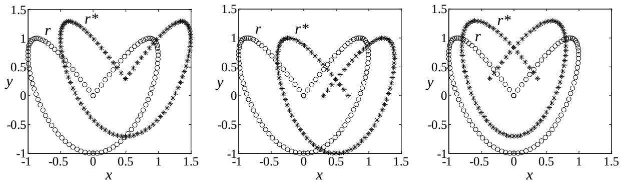
In this paragraph several examples are presented with three types of constraints: boundary, difference, and integral constraints. The last example describes an application to mechanics and two-point boundary value problems. In addition, at the end of the paragraph, a MATLAB code for the solution of one of the examples is given in the Appendix. The code could easily be adjusted to other cases.

Example 1

Consider the trajectory r defined by $\{r_i=(x_i,y_i),\ x_i=\sin(2t_i),\ y_i=\sin(3t_i),\ i=1,2,\dots,N\}$ on a uniform mesh with boundaries $a=0$, $b=\pi$, and number of mesh-points $N=101$. Using (9) and (10), the trajectory r^*, i.e. $\{r^*_i=(x^*_i,y^*_i),\ i=1,2,\dots,N\}$, similar to r and satisfying the following two integral and one difference constraints

$$\sum_{i=1}^{N}x^*_i=\sum_{i=1}^{N}x_i+\Delta S_x,\ \ \sum_{i=1}^{N}y^*_i=\sum_{i=1}^{N}y_i+\Delta S_y,\ \ x^*_k-x^*_{N-k+1}=0 \quad (11)$$

is found for several values of ΔS_x, ΔS_y, and k (see fig.1).



(a) ***(b)*** ***(c)***

Fig.1. *The original trajectory r and the similar to it trajectory r* satisfying constraints (11) for* ***(a)*** *ΔS_x=50, ΔS_y=30, k=1;* ***(b)*** *ΔS_x=50, ΔS_y=0, k=4; and* ***(c)*** *ΔS_x=0, ΔS_y=30, k=7.*

Example 2

Consider the trajectory r defined by $\{r_i=(x_i,y_i),\ x_i=t_i-2\sin(t_i),\ y_i=1-2\cos(t_i),\ i=1,2,\dots,N\ \}$ on a uniform mesh with $a=-\pi$, $b=3\pi$, and $N=101$. The trajectory r^* similar to r and satisfying the following boundary constraints

$$(x^*_1,y^*_1)=(x_1,y_1)+(\Delta x_1,\Delta y_1),$$
$$(x^*_N,y^*_N)=(x_N,y_N)+(\Delta x_N,\Delta y_N) \quad (12)$$

is found for several values of $(\Delta x_1, \Delta y_1)$ and $(\Delta x_N, \Delta y_N)$ (see fig.2).

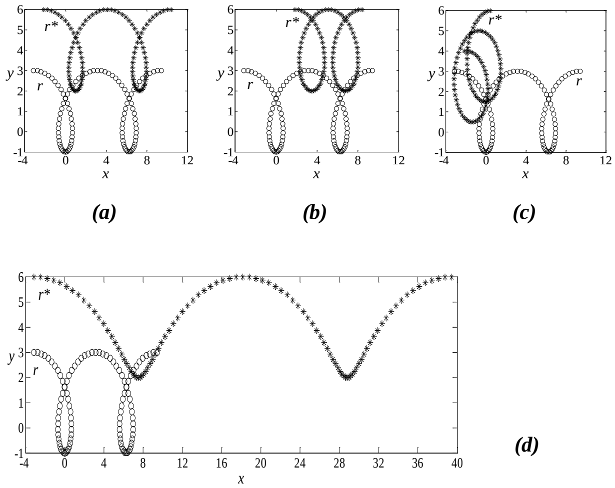


Fig.2. *The original trajectory r and the similar to it trajectory r* satisfying constraints (12) for* ***(a)*** *(Δx_1, Δy_1)=(1,3), (Δx_N, Δy_N)=(1,3);* ***(b)*** *(Δx_1, Δy_1)=(5,3), (Δx_N, Δy_N)=(−1,3) ;* ***(c)*** *(Δx_1, Δy_1)=(1,1), (Δx_N, Δy_N)=(−9,3); and* ***(d)*** *(Δx_1, Δy_1)=(0,3), (Δx_N, Δy_N)=(30,3).*

Example 3

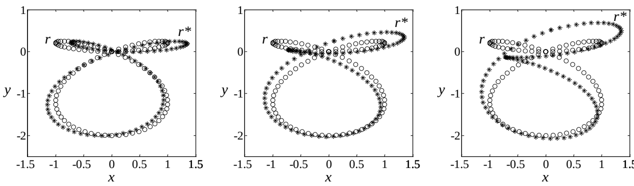
Consider the trajectory r defined by $\{r_i=(x_i,y_i),\ x_i=\sin(2t_i),\ y_i=(1-\sin(t_i))\sin(t_i),\ i=1,2,\dots,N\}$ on a uniform mesh with $a=0$, $b=2\pi$, and $N=101$. The trajectory r^* similar to r and satisfying the following four integral and two difference constraints:

$$\sum_{i=1}^{N}x^*_i=\sum_{i=1}^{N}x_i+10,\ \ \sum_{i=1}^{N}y^*_i=\sum_{i=1}^{N}y_i,$$

$$\sum_{i=1}^{N}t_ix^*_i=\sum_{i=1}^{N}t_ix_i,\ \ \sum_{i=1}^{N}(t_i-a)(t_i-b)y^*_i=\sum_{i=1}^{N}(t_i-a)(t_i-b)y_i+\Delta T_y,$$

$$x^*_1-x^*_N=0,\ \ y^*_1-y^*_N=0 \quad (13)$$

is found for several values of ΔT_y (see fig.3).



(a) ***(b)*** ***(c)***

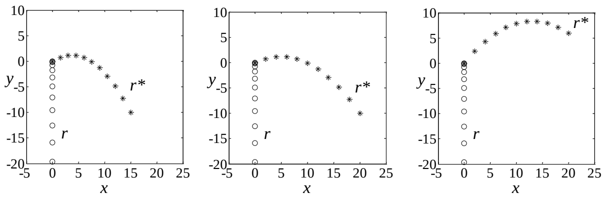
Fig.3. *The original trajectory r and the similar to it trajectory r* satisfying constraints (13) for* ***(a)*** *ΔT_y=0;* ***(b)*** *ΔT_y=50; and* ***(c)*** *ΔT_y=100.*

Example 4

A mass point, initially at rest, travels 2 seconds under the influence of the gravitational potential $U=gy$, $g=9.8$ (m/s^2). Placing the origin of the coordinate system at the initial position of the point and partitioning the time interval $t\in[0,2]$ by $N=11$ equally separated mesh-points the following discretized trajectory r is obtained: $\{r_i=(x_i,y_i),\ x_i=0,\ y_i=-gt_i^2/2,\ t_i=(i-1)h,\ i=1,2,\dots,N,\ h=0.2\ \text{(s)}\}$. The trajectory r^* similar to r and satisfying the following boundary constraints:

$$(x^*_1, y^*_1) = (0,0),\ (x^*_N, y^*_N) = (x_b, y_b), \tag{14}$$

is found for several values of (x_b,y_b) (see fig.4). The obtained trajectory r^* describes *exactly* the motion of a point travelling for 2 seconds between points (0,0) and (x_b,y_b) under the influence of the given potential. If the force field is not homogenous the trajectory r^* will describe the motion of the point only approximately. Then, however, r^* could be incorporated into a 'shooting-projection' iterative procedure to obtain the exact solution to the two-point boundary value problem [5].



(a) ***(b)*** ***(c)***

Fig.4. *The original trajectory r and the similar to it trajectory r* satisfying constraints (14) for* ***(a)*** $(x_b,y_b)=(-10,15)$ *(m);* ***(b)*** $(x_b,y_b)=(-10,20)$ *(m); and* ***(c)*** $(x_b,y_b)=(6,20)$ *(m). The coordinates x and y are measured in meters (m).*

5. Conclusion

This work defined constrained similarity of 2-D trajectories via minimizing the H^1 semi-norm of the difference between the trajectories and presented an exact solution to the discreized case. The results obtained agree with what is expected from similarity of trajectories under imposed constraints. The last example suggests possible application to mechanics and two-point boundary value problems.

6. Appendix

In this appendix a MATLAB code for solving Example 3(c) is presented. The variables `A_`, `c_`, and `L_` are used for $\bar{A}$, $\bar{c}$ and $\bar{L}$, while `xs`, `ys`, and `us` are used for x^*, y^* and u^*. The variable `l` is used for λ. To define the needed vectors and matrices, first the corresponding vectors and matrices composed of zeros and having the required size are defined.

```
function main

 N=101; M=6;
 a=0; b=2*3.141593;

 h=(b-a)/(N-1);

 t=zeros(N,1); x=zeros(N,1); y=zeros(N,1);
 u=zeros(2*N,1);
 A=zeros(M,2*N); c=zeros(M,1);
 A_=zeros(2*N,2*N); c_=zeros(2*N,1);
 L_=zeros(2*N,2*N);

 for i=1:N
  t(i)=a+(i-1)*h;
  x(i)=sin(2*t(i)); u(i)=x(i);
  y(i)=(1-sin(t(i)))*sin(t(i)); u(N+i)=y(i);
 end

 Sx=0; Sy=0; Tx=0; Ty=0;

 for i=1:N
  Sx=Sx+x(i);
  Sy=Sy+y(i);
  Tx=Tx+t(i)*x(i);
  Ty=Ty+(t(i)-a)*(t(i)-b)*y(i);
 end

 for i=1:N
  A(1,i)=1;
  A(2,N+i)=1;
  A(3,i)=t(i);
  A(4,N+i)=(t(i)-a)*(t(i)-b);
 end

 A(5,1)=1; A(5,N)=-1;
 A(6,N+1)=1; A(6,2*N)=-1;

 c(1)=Sx+10;
 c(2)=Sy;
 c(3)=Tx;
 c(4)=Ty+100;
 c(5)=0;
 c(6)=0;

 for i=1:N
  A_(1,i)=A(1,i);
  A_(N+1,N+i)=A(2,N+i);
 end

 c_(1)=c(1);
 c_(N+1)=c(2);

 L_(1,1)=-1; L_(1,2)=1;
 L_(N,N-1)=1; L_(N,N)=-1;
 L_(N+1,N+1)=-1; L_(N+1,N+2)=1;
 L_(2*N,2*N-1)=1; L_(2*N,2*N)=-1;

 for i=2:N-1
  L_(i,i-1)=1; L_(i,i)=-2; L_(i,i+1)=1;
  L_(N+i,N+i-1)=1; L_(N+i,N+i)=-2; L_(N+i,N+i+1)=1;
 end

 H=inv(L_+A_); d=A_*u-c_;

 l=(A*H*A')\(A*u-c-A*H*d)*2;
 us=u-H*(A'*l/2+d);

 for i=1:N
  xs(i)=us(i);
  ys(i)=us(N+i);
 end

 plot(x,y,'o',xs,ys,'*');
   end
```

7. References

1. Yan, Xin, *Linear Regression Analysis: Theory and Computing*, World Scientific, ISBN 9789812834119, 2009.

2. Antoniou, A., Wu-Sheng Lu, *Practical Optimization Algorithms and Engineering Applications*, Springer, 2007.

3. Filipov, S., Atanasov, A., Gospodinov, I. Constrained Functional Similarity by Minimizing the H^1 Seminorm and Applications to Engineering Problems, Journal of Chemical Technology and Metallurgy, 2016.

4. Adams, R. *Sobolev Spaces*, Academic Press, ISBN 978-0-12-044150-1, 1975.

5. Filipov, S., Gospodinov, I., Angelova, J., Solving Two-Point Boundary Value Problems for Integro-Differential Equations Using the Simple Shooting-Projection Method, Numerical Methods and Applications, pp.169-177, Springer, 2015.

6. Arfken, G. *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, 1985.